Q.1 The principal value of $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is **[1]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Solution: (a)

Q.2 If A is an invertible matrix of order 2 then $\det(A^{-1})$ is equal to **[1]**

(a) $\det(A)$ (b) $\frac{1}{\det(A)}$ (c) 1 (d) 0

Solution: (b)

Q.3 The derivative of 2^x is : **[1]**

(a) 2^x (b) $\frac{2^x}{\log 2}$ (c) $2^x \log 2$ (d) None of these

Solution: (c)

Q.4 The interval in which $y = x^2\, e^{-x}$ is increasing is **[1]**

(a) $[-\infty, \infty]$ (b) $(-2, 0)$ (c) $(2, \infty)$ (d) $(0, 2)$

Solution: (d)

Q.5 $\int e^x\,(\sin x + \cos x)\, dx$ is equal to **[1]**

(a) $e^x \cos x + c$ (b) $-e^x \sin x + c$ (c) $e^x \sin x + c$ (d) $-e^x \cos x + c$

Solution: (c)

Q.6 The degree of differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$ is **[1]**

(a) 1 (b) 2 (c) 3 (d) Not defined

Solution: (a)

Q.7 Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if : **[1]**

(a) $\vec{a}\cdot\vec{b} = 0$ (b) $\vec{a}\cdot\vec{b} \neq 0$ (c) $\vec{a}\times\vec{b} = \vec{0}$ (d) $\vec{a}\times\vec{b} \neq \vec{0}$

Solution: (c)

Q.8 **Angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively and when $\vec{a}\cdot\vec{b}=\sqrt{6}$** **[1]**

(a) 3 **(b) $\frac{\pi}{2}$** **(c) $\frac{\pi}{4}$** **(d) $\frac{\pi}{3}$**

Solution: (c)

Q.9 **If a line makes angles α, β, γ with positive direction of coordinate axes, then value of $sin^2\alpha + sin^2\beta + sin^2\gamma$ is** **[1]**

(a) -1 **(b) 2** **(c) 1** **(d) -2**

Solution: (b)

Q.10 **The probability of obtaining an even prime number on each die, when a pair of die is rolled, is :** **[1]**

(a) 0 **(b) $\frac{1}{3}$** **(c) $\frac{1}{12}$** **(d) $\frac{1}{36}$**

Solution: (d)

Q.11 **Using elementary operations, find the inverse of matrix $A=\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$** **[2]**

Solution : In order to use row operations, we may write as A = 1A

or $\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}A$

$\begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}A$ (applying $R_2 \leftrightarrow R_1$)

or $\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & -2\end{bmatrix}A$ (applying $R_2 \rightarrow R_2 - 2R_1$)

or $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}A$ (applying $R_2 \rightarrow -R_2$)

or $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}A$ (applying $R_1 \rightarrow R_1 - R_2$)

$\therefore$ $A^{-1}=\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}$

Or

For matrix

$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1 \ 3 - 6]$, **Verify that** $(AB)' = B'A'$.

Solution: Given A = $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1, 3 - 6]$

$A' = [-2 \ 4 \ 5], \quad B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

B'A' = $\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ 4 \ 5]$

B'A' = $\begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$ .....(i)

AB = $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ 3 \ - 6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

(AB)' = $\begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 12 \\ 12 & -24 & -30 \end{bmatrix}$ ....(ii)

from (i) and (ii), we get

(AB)' = B'A'

Q.12 **Examine the function given by** **[2]**

$f(x) = \begin{cases} \sin x - \cos x, x \neq 0 \\ -1, x = 0 \end{cases}$ **for continuity.**

Solution: f(x) = $\begin{cases} \sin x - \cos x \,, if \ x \neq 0 \\ -1 \quad\quad , \ if \ x = 0 \end{cases}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = -1.$

Also f(0) = -1

$\therefore \ \lim_{x \to 0} f(x) = f(0) = -1.$

Hence f(x) is continuous for all x.

Q.13 **A balloon which always remains spherical has a variabel radius. Find the rate at whcih its volume is increasing with the radius when the later is 10 cm.** **[2]**

Solution: let r be the variable radius of the balloon and v be the volume of sphere.

then volume of sphere (v) = $\frac{4}{3} \pi r^3$

Rate of change of volume w.r.t. r $= \frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$

Rate of increase in volume, $\left(\frac{dV}{dr}\right)_{r=10} = 4\pi(10)^2$

$= 400\,\pi\, cm^3/sec.$

Q.14 **Form the differential equation of the family of ellipses having foci on $y -$ axis and centre at origin.** **[2]**

Solution: let the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating w.r.t., x, we get

$$\frac{1}{a^2}.2x + \frac{1}{a^2}2y\,y_1 = 0$$

$$\Rightarrow \frac{yy_1}{b^2} = \frac{-x}{a^2}$$

$$\Rightarrow \frac{yy_1}{x} = \frac{-b^2}{a^2}$$

Again differentiating w.r.t, x, we get.

$$\frac{x\frac{d}{dx}yy_1 - yy_1}{x^2} = 0$$

$$x(yy_2 + y_1y_1) - yy_1 = 0$$

$$\Rightarrow x(y_1)^2 + xyy_2 - yy_1 = 0$$

Q.15 **If $f: R \to R$ be defined by $f(x) = x^2 - 3x + 2$, find the $f(f(x))$.** $\left[3\frac{1}{2}\right]$

Solution: Given, f(x) = $x^2 - 3x + 2$

$$\Rightarrow f(f(x)) = f(x^2 - 3x + 2)$$

$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= x^4 + 9x^2 + 4 - 6x^3 + 4x^2 - 12x - 3x^2 + 9x - 6 + 2$$

$$= x^4 - 6x^3 + 10x^2 - 3x.$$

Q.16 **Solve $tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\, tan^{-1}x, (x > 0)$** $\left[3\frac{1}{2}\right]$

Solution: Given, $tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\, tan^{-1}x; x > 0.$

$$\Rightarrow tan^{-1}x = 2\, tan^{-1}\frac{1-x}{1+x} = tan^{-1}\left\{\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right\}$$

$\Rightarrow x = \frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2} = \frac{2(1-x^2)}{4x} = \frac{1-x^2}{2x}$

$\Rightarrow 2x^2 + x^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow x = \frac{1}{\sqrt{3}}$

(because, given that x > 0)

Or

Express the following in the simplest form:

$\boldsymbol{tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0}$

Solution: $\boldsymbol{tan^{-1}\frac{\sqrt{1+x^2-1}}{x}, x \neq 0}$

Put $x = tan\theta \quad \Rightarrow \theta = tan^{-1}x$

$= tan^{-1}\frac{\sqrt{1+tan^2\theta-1}}{\tan\theta}, x \neq 0$

$= tan^{-1}\left(\frac{sec\theta-1}{tan\theta}\right)$

$= tan^{-1}\left(\frac{\frac{1}{cos\theta}-1}{\frac{sin\theta}{cos\theta}}\right)$

$= tan^{-1}\left(\frac{1-cos\theta}{cos\theta} \times \frac{cos\theta}{sin\theta}\right)$

$= tan^{-1}\left(\frac{1-cos\theta}{sin\theta}\right)$

$= tan^{-1}\left(\frac{2sin^2\frac{\theta}{2}}{2sin\frac{\theta}{2}cos\frac{\theta}{2}}\right)$

$= tan^{-1}\left(\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}\right)$

$= tan^{-1}\left(tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2}$

$= \frac{1}{2}\, tan^{-1}x\,.$

Q.17 **Prove that** $\boldsymbol{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)}$ $\left[3\frac{1}{2}\right]$

Solution: Consider, L.H.S. = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Operating $R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1, we\ get$

$$\text{L.H.S.}=\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 1 & c+a\end{vmatrix}$$

(taking (b-a) fromR_2 *and* $(c-a)$ *from* R_3 *common*)

$\Rightarrow$ L. H. S. $=(\mathrm{b}-\mathrm{a})(\mathrm{c}-\mathrm{a})(\mathrm{c}+\mathrm{a}-\mathrm{b}-\mathrm{a})$

$=(b-a)(c-a)(c-a)$

$=(a-b)(b-c)(c-a)=$ *R.H.S. Hence proved.*

Q.18 **Differentiate $\cos(\log x+e^x)$ w.r.t. x** $\left[3\frac{1}{2}\right]$

Solution: Let y= cos(logx+e^x), $x>0$

Then $\frac{dy}{dx}=-\sin(\log x+e^x)\cdot\frac{d}{dx}(logx+e^x)$.

$=-\sin(logx+e^x)\cdot\left(\frac{1}{x}+e^x\right)$

$=\frac{-(1+xe^x)\sin(logx+e^x)}{x}$.

Or

Find $\frac{dy}{dx}$ if $(\cos x)^y=(\cos y)^x$

Solution: (i) Given that $(cos\ x)^y=(\cos y)^x$

Taking logarithm on both sides, we have

y log (cos x) = x log (cos y)

Differentiating both sides w.r.t, x, we have

$y\cdot\frac{d}{dx}(\log(\cos x)+\log(\cos x)\cdot\frac{dy}{dx}$

$=x\cdot\frac{d}{dx}\log(\cos y)+\log(\cos y)\cdot 1$

$\Rightarrow y\cdot\frac{1}{\cos x}(-\sin x)+\log(\cos x)\cdot\frac{dy}{dx}$

$=x\cdot\frac{1}{\cos y}(-\sin y)\cdot\frac{dy}{dx}+\log(\cos y)$

$\Rightarrow -y\tan x+\log(\cos x)\frac{dy}{dx}=-x\tan y\frac{dy}{dx}+\log(\cos y)$

$\Rightarrow \frac{dy}{dx}(x\tan y+\log(\cos y))=y\tan x+\log(\cos y)$

$\Rightarrow \frac{dy}{dx}=\frac{y\tan x+\log(\cos y)}{x\tan y+\log(\cos x)}$.

Q.19 **Evaluate $\int \frac{x}{(x^2+1)(x-1)}\,dx$** $\left[3\frac{1}{2}\right]$

Solution: $I = \int \frac{x}{(x^2+1)(x-1)}dx$

by using Partial fractions, we have

$$\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{c}{x-1}$$

$$\frac{x}{(x^2+1)(x-1)} = \frac{(Ax+B)(x-1)+c(x^2+1)}{(x^2+1)(x-1)}$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

$$x = (A+C)x^2 + (-A+B)x - B + C$$

$$0x^2 + x + 0 = (A+C)x^2 + (-A+B)x + (-B+C)$$

On comparing coefficients, we have

$A + C = 0,$ $\quad -A + B = 1,$ $\quad -B + C = 0$

$C = -A,$ $\quad C + C = 1$ $\quad B = c$

$2C = 1$

$\therefore \boldsymbol{A = -\frac{1}{2}, B = \frac{1}{2}}$ $\quad C = 1$

$$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}$$

$$\therefore I = \int \frac{x}{(x^2+1)(x-1)}dx = \left(-\frac{1}{2}\right)\int \frac{x-1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= \left(-\frac{1}{2}\right)\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\log|x^2+1| + \frac{1}{2}tan^{-1}x + \frac{1}{2}\log|x-1| + c$$

Q.20 **Evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$** $\left[3\frac{1}{2}\right]$

Solution: $I = \int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Let $x + 2 = A\,(2x+2) + B$

$\Rightarrow x + 2 = 2Ax + 2A + B$

On comparing coefficients, we have

$2A = 1$ $\quad 2A + B = 2 \Rightarrow 2\cdot\frac{1}{2} + B = 2$

$\Rightarrow A = \frac{1}{2}$ $\quad \Rightarrow 1 + B = 2$

$\Rightarrow \quad B = 1$

$\therefore x + 2 = \frac{1}{2}(2x+2) + 1$

$$l = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int (x^2+2x+3)^{-1/2}(2x+2)dx + \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$$

$$= \frac{1}{2}\frac{(x^2+2x+3)^{1/2}}{1/2} + \log\left|x+1+\sqrt{x^2+2x+3}\right| + c$$

$$= \sqrt{x^2+2x+3} + \log\left|x+1=\sqrt{x^2+2x+3}\right| + c \ .$$

Q.21 **Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}$ dx** $\left[3\frac{1}{2}\right]$

Solution: Let $l = \int_0^{\pi} \frac{x}{1+\sin x}dx$ ....(i)

using, $\int_0^a f(x)dx = \int_0^a f(a-x)dx, we\ get$

$$l = \int_0^{\pi} \frac{(\pi-x)}{1+\sin(\pi-x)}dx = \int_0^{\pi} \frac{(\pi-x)}{1+\sin x}dx \quad \text{....(ii)}$$

$[because, \sin(\pi-\theta) = \sin\theta]$

Adding equation (i) and (ii) , we get

$$21 = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}dx \Rightarrow 21 = \pi\int_0^{\pi} \frac{dx}{1+\sin x}$$

$$2I = \int_0^{\pi} \frac{\pi}{(1+\sin x)}$$

$$2I = \pi\int_0^{\pi} \frac{1-\sin x}{1-sin^2 x} = \pi\int_0^{\pi} \frac{1-\sin x}{cos^2 x} = \pi\int_0^{\pi} \frac{1-\sin x}{cos^2 x}$$

$$2I = \pi\left[\tan x - \sec x\right]_0^{\pi} = \pi\left[(0-(-1))-(0-1)\right] = \pi\left[1+1\right] = 2\pi$$

$$I = \pi$$

$\Rightarrow \int_0^{\pi} \frac{x}{1+\sin x} dx = \pi.$

Q.22 **Solve the differential equation :** $\left[3\frac{1}{2}\right]$

$$y\,dx + (x - y^2)dy = 0$$

Solution: $\mathrm{y\,dx + (x - y^2)dy = 0}$

$\Rightarrow y\,dx = (y^2 - x)dy$

$\Rightarrow dx = \left(\frac{y^2-x}{y}\right)dy$

$\Rightarrow \frac{dx}{dy} = y - \frac{1}{y}\cdot x \quad \Rightarrow \quad \frac{dx}{dy} + \frac{1}{y}\cdot x = y$

This is of form, $\frac{dy}{dy} + Px = Q, we\ have$

$P = \frac{1}{y,} Q = y$

I.F.$= e^{\int P\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$

Therefore, the general solution is

$x.y = \int y.y\,dy + c$

$x.y = \frac{y^3}{3} + c$

Or

Solve the differential equation :

$$(x-y)\frac{dy}{dx} = x + 2y$$

Solution: The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

Let $F(x,y) = \frac{x+2y}{x-y}$

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right)$$

R.H.S of differential equataion (2) is of the form g$\left(\frac{y}{x}\right)$ amd so it is a homogenous function of degree zero. Therefore, equation (i) is a homogeneous differential equation. To solve it we make the substitution

$$\mathrm{y = vx}$$

Differentiating equation (3) with respect to , x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or $$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or $$x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

or $$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int\frac{v-1}{v^2+v+1}dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\int\frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

Or $$\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv - \frac{3}{2}\int\frac{1}{v^2+v+1}dv = -log|x| + C_1$$

Or $$\frac{1}{2}log|v^2+v+1| - \frac{3}{2}\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv = -log|x| + C_1$$

Or $$\frac{1}{2}log|v^2+v+1| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -log|x| + C_1$$

Or $$\frac{1}{2}log|v^2+v+1| + \frac{1}{2}logx^2 = \sqrt{3}\,tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_1$$

Replacing v by $\frac{y}{x}$, *we get*

Or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| + \frac{1}{2}logx^2 = \sqrt{3}\ tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C_1$$

Or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right| = \sqrt{3}\ tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C_1$$

Or $$\log|(y^2+xy+x^2)| = 2\sqrt{3}\,tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + 2C_1$$

Or $$\log|(x^2+xy+y^2)| = 2\sqrt{3}\,tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C$$

Which is the general solution of the differential equation (1).

Q.23 **Find λ if the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{c} = \hat{\imath} + \lambda\hat{\jmath} - 3\hat{k}$ are coplanar.** $\left[3\frac{1}{2}\right]$

Solution: $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix}1 & -1 & 1\\ 3 & 1 & 2\\ 1 & \lambda & -3\end{vmatrix} = 0$ $[\because \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar$]$

$$\Rightarrow 1(-3-2\lambda) + 1(-9-2) + 1(3\lambda-1) = 0$$

$\Rightarrow \quad (-3-9-1)-(2\lambda-3\lambda)=0$

$\Rightarrow \quad -13+\lambda=0$

$\Rightarrow \quad \lambda=13$

Q.24 Find the vector and Cartesian equations of plane that passes through the point $(1,4,6)$ and the normal vector to the plane is $\hat{\imath}-2\hat{\jmath}+\hat{k}$. $\left[3\frac{1}{2}\right]$

Solution: The equation of plane which passes through the point (1,4,6) and normal vector to plane is $\hat{\imath}-\widehat{2}j+\hat{k}$ given by

$$[\vec{r}-(\hat{\imath}+4\hat{\jmath}+6\hat{k})]\cdot[\hat{\imath}-2\hat{\jmath}+\hat{k}]=0$$

$\Rightarrow \quad [(x-1)\hat{\imath}+(y-4)\hat{\jmath}+(z-6)\hat{k}]:[\hat{\imath}-2\hat{\jmath}+\hat{k}]=0$

$\Rightarrow \quad x-1-2y+8+z-6=0$

$\Rightarrow \quad x-2y+z+1=0$

Q.25 A fair coin and unbiased die are tossed. Let A be the event 'head appears on the coin' on the event 'head appears on the coin' and B be the even '3 on the die'. Check whether A and B are independent events or not. $\left[3\frac{1}{2}\right]$

Solution: Sample space = $\{(H,1),(H,2),\ldots\ldots(H,6)\ (T,1),(T,2),\ldots\ldots.(T,6)\}$

$A=\{(H,1),(H,2),\ldots\ldots(H,6)\}$

$B=\{(H,3),(T,3)\}$

$A\cap B=\{(H,3)\}$

$P(A)=\frac{6}{12}=\frac{1}{2}$

$P(B)=\frac{2}{12}=\frac{1}{6}$

$P(A\cap B)=\frac{1}{12}$

$P(A)\times P(B)=\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}.$

$\therefore P(A\cap B)=P(A)\times P(B).$

Hence, A and B are independent events.

Q.26 If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) at least six heads

(iii) at most six heads

Solution : Let P(a success) $= \frac{1}{2},\ P(failure) = 1 - \frac{1}{2} = \frac{1}{2}$.

(i) P (exactly six heads)P (X=6)

$$= {}^{10}C_6\, p^6 q^4 = {}^{10}C_6 \cdot \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 = \frac{105}{256}$$

(ii) $P(atleast\ six\ heads) = p(X \geq 6)$

=P(6) +P(7) +P(8) +P(9) +P(10)

$$= {}^{10}C_6 \left(\frac{1}{2}\right)^{10} + {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} \left[\frac{10!}{6!4!} + \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!}\right] = \frac{193}{512}$$

(iii) P(atmost six heads)

$$= \left(\frac{1}{2}\right)^{10} \cdot + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + {}^{10}C_2 \left(\frac{1}{2}\right)^{10} + {}^{10}C_3 \left(\frac{1}{2}\right)^{10} + {}^{10}C_4 \left(\frac{1}{2}\right)^{10} + {}^{10}C_5 \left(\frac{1}{2}\right)^{10} + {}^{10}C_6 \left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}.$$

or

If A and B are two events such that P(A) $= \frac{1}{4}$, P(B)$= \frac{1}{2}$ and P(A$\cap B) = \frac{1}{8}$, find P (not A and not B).

Solution : Since P(A$\cap B) = \frac{1}{8} = \frac{1}{4} \times \frac{1}{2} = P(A).P(B)$

Therefore, A and B are independent implies that A^c *and* B^c are also independent.

Therefore, P$(A^c \cap B^c) = P(A^c)P(B^c)$

$\Rightarrow P(not\ A\ and\ not\ B) = P(A^c \cap B^c) = P(A^c)\ P(B^c)$

= (1-P(A)(1-P(B)

$= \left(1 - \frac{1}{4}\right)\ \left(1 - \frac{1}{2}\right) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}.$

Q. 27 **Solve the following equations by Matrix method:** **[5]**

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Solution : Given system of equations is

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$\mathbf{2x - y + 3z = 12}$

It can be written as AX=B

Where, A=$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$

$= 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 4 \neq 0$

Since $|A| \neq 0, therefore\ A^{-1} exists.$

Now, adj A = $\begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$

And $A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow x = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$

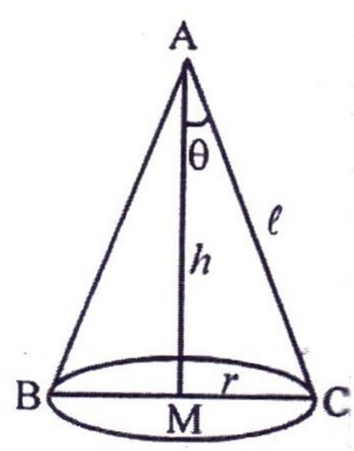
$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Therefore, x =2, y = 1, z = 3.

Q.28 **Show that the semi-vertical angle of right circular cone of given surface area and maximum Volume is $sin^{-1}\left(\frac{1}{3}\right)$.** **[5]**

Solution : We know that,

Total surface area, $S = \pi r l + \pi r^2$



$\Rightarrow S = \pi r\sqrt{r^2 + h^2} + \pi r^2$

(because in $\Delta CMA\ l^2 = h^2 + r^2$)

$\Rightarrow S - \pi r^2 = \pi r\sqrt{r^2 + h^2}$

$\Rightarrow S^2 + \pi^2 r^4 - 2S\pi r^2 = \pi^2 r^2(r^2 + h^2)$

$\Rightarrow S^2 + \pi^2 r^4 - 2S\pi r^2 = \pi^2 r^4 + \pi^2 r^2 h^2$

$\Rightarrow S^2 - 2\, S\pi r^2 = \pi^2 r^2 h^2$

$\Rightarrow h^2 = \frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi} \Rightarrow h = \sqrt{\frac{S^2}{\pi^2 r^2} - \frac{2s}{\pi}}$ ......(i)

Also, volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{\frac{S^2}{\pi^2 r^2} - \frac{2s}{\pi}}$

Since V is maximum iff V^2 is maximum, therefore to avoid lengthy calculations, we will find Maximum value of V^2.

Now, $V^2 = \frac{1}{9}\pi^2 r^4 \left(\frac{S^2}{\pi^2 r^2} - \frac{2s}{\pi}\right)$

$\Rightarrow V^2 = \frac{S^2 r^2}{9} - \frac{2S\pi r^4}{9}$

$\Rightarrow \frac{d}{dr}(V^2) = \frac{2rS^2}{9} - \frac{8r^3 S\pi}{9}$ ....(ii)

$\Rightarrow \frac{d^2}{dr^2}(V^2) = \frac{2S^2}{9} - \frac{24r^2 S\pi}{9}$ ...(iii)

For maxima or minima, $\frac{d}{dr}(V^2) = 0$, then from (ii), we have

$\frac{2rS^2}{9} - \frac{8r^3 S\pi}{9} = 0 \Rightarrow S = 4\pi r^2 \Rightarrow r = \sqrt{\frac{S}{4\pi}}$

Now, from, (iii), $\left.\frac{d^2}{dr^2}(V^2)\right]_{r=\sqrt{\frac{S}{4\pi}}} < 0$

Thus V^2 is maximum and hence V is maximum, when $r = \sqrt{\frac{S}{4\pi}}$,

then from (i), we have

$h = \sqrt{\frac{S^2.4\pi}{\pi^2 S} - \frac{2S}{\pi}} = \sqrt{\frac{2S}{\pi}}$

Let θ is the semi-vertical angle of the cone when volume is maximum, therefore from

$\Delta CMA, \sin\theta = \frac{r}{l} = \frac{r}{\sqrt{r^2+h^2}}$

$\Rightarrow \sin\theta = \frac{\sqrt{\frac{S}{4\pi}}}{\sqrt{\frac{S}{4\pi}+\frac{2S}{\pi}}} = \frac{1}{\sqrt{1+8}} = \frac{1}{3} \Rightarrow \theta = sin^{-1}\frac{1}{3}$. Hence proved

or

Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$.

Solution: $y = (x-2)^2$

$\frac{dy}{dx} = 2(x-2)$

Slope of tangent $= 2x - 4$

The chord passes through $P(2,0)$ and $Q(4,4)$.

slope of chord $= \frac{4-0}{4-2} = \frac{4}{2} = 2.$

$\because$ tangent is parallel to chord.

$\therefore\ 2x - 4 = 2$

$\Rightarrow 2x = 2 + 4$

$\Rightarrow 2x = 6$

$\Rightarrow x = 3$

When $x = 3, y = (3-2)^2 = 1$

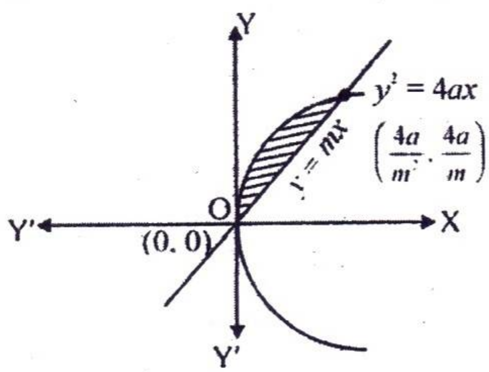
$\therefore$ Point $(3, 1)$.

Q.29 Find the area of region bounded by the curves : **[5]**

$y^2 = 4ax$ and the line $y = mx$.

Solution: The given curves are $y^2 = 4ax$ ....(i)

and $y = mx$ ....(ii)



Also, the line and the curve meets at the point $(0, 0)$ and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$

Therefore, the required area = are of shaded region

$= \int_0^{4a/m^2} y\,dx - \int_0^{\frac{4a^2}{m}} y\,dx = \int_0^{\frac{4a^2}{m}} \left(\sqrt{4\,ax} - mx\right)dx$

$= \left[\frac{\sqrt{4a}x^{3/2}}{3/2}\right]_0^{4a/m^2} - \left[\frac{mx^2}{2}\right]_0^{4a/m^2}$

$= \frac{2}{3}\sqrt{4a}\,\frac{(4a)^{3/2}}{m^3} - \frac{m(4a)^2}{2m^4} = \frac{2}{3}\,\frac{(4a)^2}{m^3} - \frac{(4a)^2}{2m^3}$

$= \frac{32a^2}{3m^2} - \frac{16a^2}{2m^3} = \frac{32a^2}{3m^2} - \frac{8a^2}{m^3} = \frac{32a^2 - 24a^2}{3m^3}$

$= \frac{8a^2}{3m^3}$ square units.

Or

Using integration find the are of region bounded by the lines $2x + y = 4, 3x - 2y = 6$ and $x - 3y + 5 = 0$.

Solution: The given lines are

$2x + y = 4$ ....(1)

$3x - 2y = 6$ ....(2)

$x - 3y + 5 = 0$ ....(3)

On solving (1) and (2);

$2x + y = 4 \Rightarrow 4x + 2y = 8$

$3x - 2y = 6 \Rightarrow 3x - 2y = 6$

$7x = 14 \Rightarrow x = 2$

$\therefore x = 2 \Rightarrow 4 + y = 4 \Rightarrow y = 0$

$\therefore A(2,0)$

On solving (2) and (3) :

$3x - 2y = 6 \Rightarrow 3x - 2y = 6$

$x - 3y = -5 \Rightarrow +3x - 9y = -15$

$7y = 21 \Rightarrow y = 3$

$\therefore 3x - 2 \times 3 = 6$

$\therefore B(4,3).$

On solving (1) and (3):

$2x + y = 4 \Rightarrow 2x + y = 4$

$x - 3y = -5 \Rightarrow 2x - 6y = -10$

$7y = 14$

$y = 2$

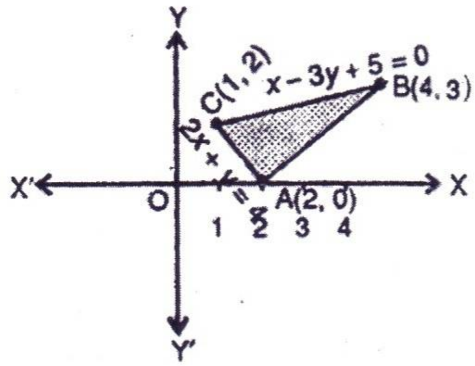
When $y = 2, 2x + 2 = 4$

$\Rightarrow 2x = 2 \Rightarrow x = 1$

$C(1,2).$

The Rough sketch of the graphs.

Area of ΔABC



$= \int_1^4 \left(\frac{x+5}{3}\right) dx$

$- \int_1^2 (4-2x)dx - \int_2^4 \left(\frac{3x-6}{2}\right) dx$

$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - \left[4x - \frac{2x^2}{2}\right]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$

$= \frac{1}{3}\left[(8+20) - \left(\frac{1}{2}+5\right)\right] - [(8-4)-(4-1)]$

$- \frac{1}{2}[(24-24)-(6-12)]$

$= \frac{1}{3}\left[28 - \frac{11}{2}\right] - [4-3] - \frac{1}{2}[6]$

$= \frac{1}{3}\left[\frac{56-11}{2}\right] - 1 - 3$

$= \frac{1}{3} \times \frac{45}{2} - 4$

$= \frac{15}{2} - \frac{4}{1} = \frac{15-8}{2} = \frac{7}{2}$ sq. units.

Q.30 Find the shortest distance between the lines **[5]**

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})$ and

$\vec{r} = (4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}) + \mu(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$

Solution: Given, equations of lines are

$\vec{r} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} + \lambda(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})$ ....(i)

and $\vec{r} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k} + \mu(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$ ....(ii)

Comparing equation (i) and (ii) with $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and

$\vec{r} = \overrightarrow{a_2} + \lambda\overrightarrow{b_2}$ respectively, we get

$\overrightarrow{a_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \overrightarrow{b_1} = \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

and $\overrightarrow{a_2} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}, \overrightarrow{b_2} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

Now, $\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{\imath} - (1-4)\hat{\jmath} + (3+6)\hat{k}$

$\Rightarrow \overrightarrow{b_1} \times \overrightarrow{b_2} = -9\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$

$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$

Also, $\overrightarrow{a_2} - \overrightarrow{a_1} = (4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$\Rightarrow \overrightarrow{a_2} - \overrightarrow{a_1} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$

Therefore, the shortest distance between given lines is

$$d = \left|\frac{(\overrightarrow{b_1} \times \overrightarrow{b_2}).(\overrightarrow{a_2} - \overrightarrow{a_1})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}\right|$$

$$\Rightarrow \quad d = \left|\frac{(-9\hat{\imath}+3\hat{\jmath}+9\hat{k}).(3\hat{\imath}+3\hat{\jmath}+3\hat{k}}{3\sqrt{19}}\right|$$

$$\Rightarrow \quad d = \left|\frac{-27+9+27}{3\sqrt{19}}\right| = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}} \text{ units.}$$

Or

Find the equation of the plane that passes through three points(1,1,0), (1,2,1) and (-2,2,-1).

Solution: Let the equation of plane which passes through the point P(1,1,0) is

$$A(x-1) + B(y-1) + C(z-0) = 0$$

$\Rightarrow A(x-1) + B(y-1) + C(z-1) = 0$ ......(i)

Equation (1) passes through Q(1,2,1) is

$$A(1-1) + B(2-1) + C \cdot 1 = 0$$

$\Rightarrow 0A + B + C = 0$ ....(ii)

Equation (1) also passes through R(-2,2,-1)

$$A(-2-1) + B(2-1) + C(-1) = 0$$

$$-3A + B - C = 0$$

By eliminating A, B, C, from equation (1), (2) and (3),

$$\begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(-1-1) - (y-1)(0+3) + z(0+3) = 0$

$\Rightarrow -2x + 2 - 3y + 3 + 3z = 0$

$\Rightarrow -2x - 3y + 3z + 5 = 0$

$\Rightarrow \quad 2x + 3y - 3z - 5 = 0.$

Q.31 **Solve the following linear programming problem graphically.** **[5]**

Maximize $Z = 3x + 2y$

subject to the following constraints:

$$X + 2y \leq 10,$$

$$3x + y \leq 15,$$

$x \geq 0, y \geq 0$

Solution: 1st of all we shall plot the graph of following equation

X+2y=10

3x+y=15

X=0

Y=0

For, x+2y=10

$AT\ X-\ axis, y = 0 \Rightarrow x = 10\ A\ (10,0)$

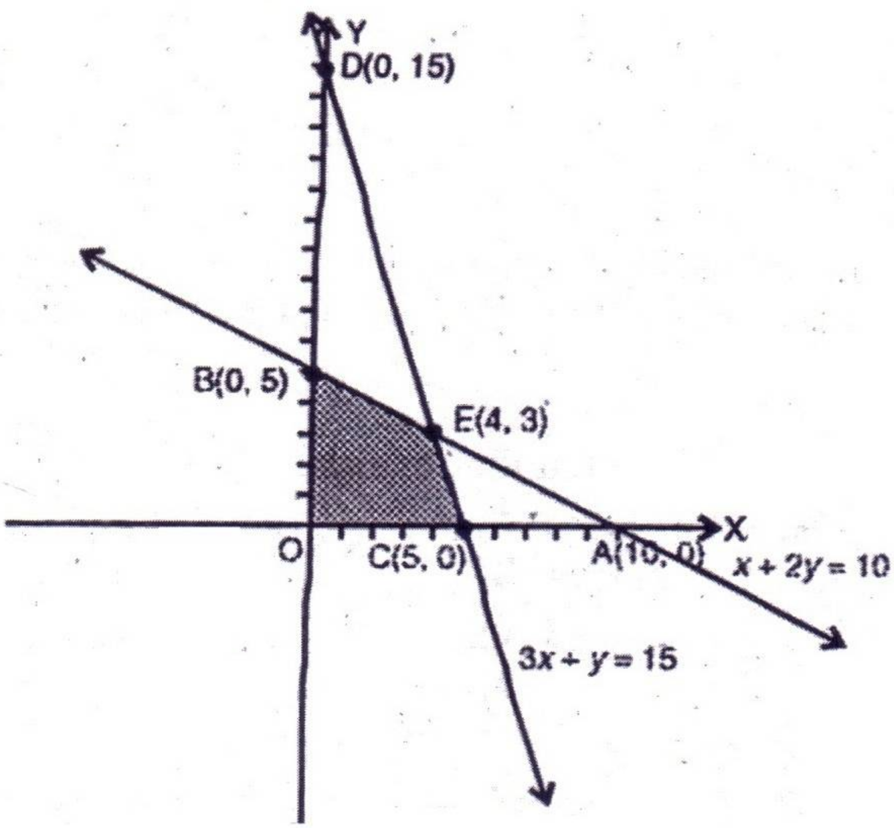
$At\ Y-axis, x = 0 \Rightarrow y = 5 \therefore D(0,5)$

For $3x + y = 15$

$At\ X - axis\ y = 0 \Rightarrow y = 5 \therefore c = (5,0)$

$At\ Y - axisx = 0 \Rightarrow 15 \therefore D(0,15)$

The feasible solution is OCEB.



Where E is intersection point of $x + 2y = 10$

and $3x + y = 15$

$\therefore E \leftrightarrow (4,3)$

$At\ C(5,0)\ Z = 15$

$At\ B(0,5), Z = 0 + 2 \times 5 = 10$

$At\ E(4,3), Z = 3 \times 4 + 2 \times 3 = 12 + 6 = 18$

$\therefore Z = 18\ is\ the\ max. value\ at\ E \leftrightarrow (4,3).$